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TESTS OF TWO CONVECTION THEORIES FOR RED GIANT AND RED SUPERGIANT ENVELOPES

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ABSTRACT

Two theories of stellar envelope convection are considered here in the context of red giants and red supergiants of intermediate to high mass: Böhm-Vitense's standard mixing-length theory (MLT) and Canuto & Mazzitelli's new theory incorporating the full spectrum of turbulence (FST). Both theories assume incompressible convection. Two formulations of the convective mixing length are also evaluated: l proportional to the local pressure scale height (H_P) and l proportional to the distance from the upper boundary of the convection zone (z). Applications to test both theories are made by calculating stellar evolutionary sequences into the red phase of core helium burning. Since the theoretically predicted effective temperatures for cool stars are known to be sensitive to the assigned value of the mixing length, this quantity has been individually calibrated for each evolutionary sequence. The calibration is done in a composite Hertzsprung-Russell diagram for the red giant and red supergiant members of well-observed Galactic open clusters. The MLT model requires the constant of proportionality for the convective mixing length to vary by a small but statistically significant amount with stellar mass, whereas the FST model succeeds in all cases with the mixing length simply set equal to z (as Canuto & Mazzitelli have also found for the Sun and low-mass red giants). The structure of the deep stellar interior, however, remains very nearly unaffected by the choices of convection theory and mixing length. Inside the convective envelope itself, a density inversion always occurs, but is somewhat smaller for the convectively more efficient MLT model. On physical grounds the FST model is preferable, and seems to alleviate the problem of finding the proper mixing length.

Subject headings: convection — stars: evolution — stars: interiors — stars: late-type — supergiants

1. INTRODUCTION

Convection in stellar envelopes is difficult to calculate accurately because of its turbulent character and its nonadiabatic behavior near the surface. Standard mixing-length theory (MLT), in the version presented by Böhm-Vitense (1958) and others, has been used for many years as a simple computational tool to treat stellar convection, with at least some laboratory, field, and other theoretical support behind it. Its primary drawback as a theoretical model of stellar turbulence is that it computes all convective quantities under the incomplete assumption of a large "average" eddy traveling a fixed distance (the mixing length) before losing its identity. This simplifying assumption introduces into the description of convection a number of poorly constrained free parameters, whose uncertainties are mitigated only by the fortunate circumstance that they can be subsumed, in many stellar applications, by the uncertainty of the unknown mixing length itself (e.g., Henyey, Vardya, & Bodenheimer 1965; Pedersen, VandenBerg, & Irwin 1990). Numerous modifications, extensions, and replacements of MLT have been proposed in the literature (Spruit, Nordlund, & Title 1990), but, being incomplete theories, they likewise contain at least one free parameter, and so enjoy no real advantage over MLT. A more serious obstacle to using these theories is that most are difficult to implement in a stellar-evolution code.

A recently proposed alternative to MLT is the easily implemented model derived by Canuto & Mazzitelli (1991, 1992). These authors computed the full spectrum of sizes of turbulent eddies (which can be a very broad spectrum for the nearly inviscid stellar plasma), and they also calculated size anisotropy without assuming its magnitude. The much sounder physical foundation of their "full spectrum theory" (FST) as compared to MLT is enhanced by its economy of free parameters: there is only one free parameter—the mixing length of the largest eddy. Nevertheless, both theories of stellar convection are simple to use and are, in effect, mainly dependent on only one unknown parameter.

Standard calibrations of the mixing length have traditionally utilized the Sun or some other solar-type star. No obvious astrophysical contradictions to MLT have arisen in any detailed applications, like those made specifically to the Sun, Population I lower main-sequence stars, Population II main-sequence subdwarfs, and old red giants of both populations (e.g., VandenBerg 1983, 1985, 1992; VandenBerg & Bridges 1984; Bergbusch & VandenBerg 1992). On the other hand, FST passes all of the same observational tests at least as well as MLT does (Canuto & Mazzitelli 1991; D'Antona, Mazzitelli, & Gratton 1992; Paternò et al. 1993; D'Antona & Mazzitelli 1994; Basu & Antia 1994).

The most useful nonsolar tests have so far concentrated on red giants in a relatively narrow mass range, roughly 0.8–2 $M_\odot$. It is therefore desirable to extend these tests and comparisons of the two convection theories to even more massive stars. Here we focus on red giants and red supergiants of intermediate to high mass, in which the star's outer envelope structure is fairly sensitive to the mixing length adopted. Consistency of an adopted stellar convection theory can at present be assessed only through consideration of the mixing length that is needed to achieve agreement with observations of effective temperatures over a wide range of stellar luminosities.

In § 2, the basic equations for the two convection theories are expressed in very simple forms that illustrate their similarities and differences in a way that supplements the discussion by Canuto & Mazzitelli (1991). Possible choices for the convective mixing length are discussed in § 3. Other physical and observational data used are described in §§ 4 and 5. Results derived from our new evolutionary sequences of stellar models are presented and compared with observations in § 6, while § 7 summarizes our main conclusions.

2. BASIC EQUATIONS FOR THE TWO CONVECTION THEORIES

In the standard MLT model, the basic equation to be solved (eq. [14.82] of Cox & Giuli 1968) can be written

$$\Sigma + \tfrac{1}{2}a_0[(1 + \Sigma)^{1/2} - 1]^3 - D = 0 \,, \tag{1}$$

where

$$\Sigma = 4A^2(\nabla - \nabla_{\rm ad}) \,, \quad D = 4A^2(\nabla_{\rm rad} - \nabla_{\rm ad}) \,, \tag{2}$$

$$\nabla = d\ln T/d\ln P \,, \quad a_0 = 9/4 \,, \tag{3}$$

$$A = Q^{1/2}(l^2/9\chi)(g/2H_P)^{1/2} \,, \tag{4}$$

$$Q = (4 - 3\beta)/\beta - (\partial \ln \mu/\partial \ln T)_P \,. \tag{5}$$

Here l is the mixing length, χ is the thermometric conductivity, g is the gravitational acceleration, H_P is the local pressure scale height, β is the ratio of gas pressure to total pressure, and μ is the mean molecular weight of the gas. The expression given for Q takes account of radiation pressure (Stothers 1963, p. 20), which was not included in Böhm-Vitense's (1958) original theory. Equation (1) is a simple cubic equation for $(1 + \Sigma)^{1/2}$, from which $\nabla - \nabla_{\rm ad}$ follows. Quick inspection yields the two limiting solutions: $\Sigma \rightarrow D$ as $D \rightarrow 0$, and $\Sigma \rightarrow (2a_0^{-1}D)^{2/3} \approx 0.9245D^{2/3}$ as $D \rightarrow \infty$. The cubic form of the basic equation (Temesváry 1960) and the two general asymptotic behaviors (Gough & Weiss 1976) are already well known.

In the FST model proposed by Canuto & Mazzitelli (1991), the analogous equation to be solved (their eq. [36]) can be written

$$\Sigma + a_1\Sigma^{m+1}[(1 + a_2\Sigma)^n - 1]^p - D = 0 \,, \tag{6}$$

where $a_1 = 24.868$, $a_2 = 9.7666 \times 10^{-2}$, $m = 0.14972$, $n = 0.18931$, and $p = 1.8503$. The three exponents m, n, and p were fitted by Canuto & Mazzitelli to numerical solutions for incompressible convection, so as to combine into the exponents of the MLT cubic equation in the limit $\Sigma \rightarrow \infty$, where the two models converge (apart from a numerical constant). Canuto & Mazzitelli did not solve their basic equation (36) directly, but solved an equivalent set of equations iteratively in order to find $\nabla - \nabla_{\rm ad}$ as a function of A and $\nabla_{\rm rad} - \nabla_{\rm ad}$. They presented their numerical results in a three-dimensional table. We note that they included the Q factor in the definition of A only in a subsequent paper (Canuto & Mazzitelli 1992) and that their numerical value of a_1, which contains the Kolmogorov constant, *may* be too small by a factor of ~ 1.7 (D'Antona & Mazzitelli 1994).

We have not formally used their table, but have solved equation (6) directly, using an iterative method. Notice that, for both small D and large D, equation (6) goes over into a simple cubic equation for Σ. In the two asymptotic limits: $\Sigma \rightarrow D$ as $D \rightarrow 0$, and $\Sigma \rightarrow (a_1^{-1}a_2^{-np}D)^{2/3} \approx 0.2021D^{2/3}$ as $D \rightarrow \infty$.

The general solution, Σ as a function of D, is plotted for both convection theories in Figure 1. The two solutions diverge appreciably only for $D > 10^3$, and then attain a difference equal to a constant factor of about 4.5752. Large D corresponds to convective layers that are either very deep (large A)

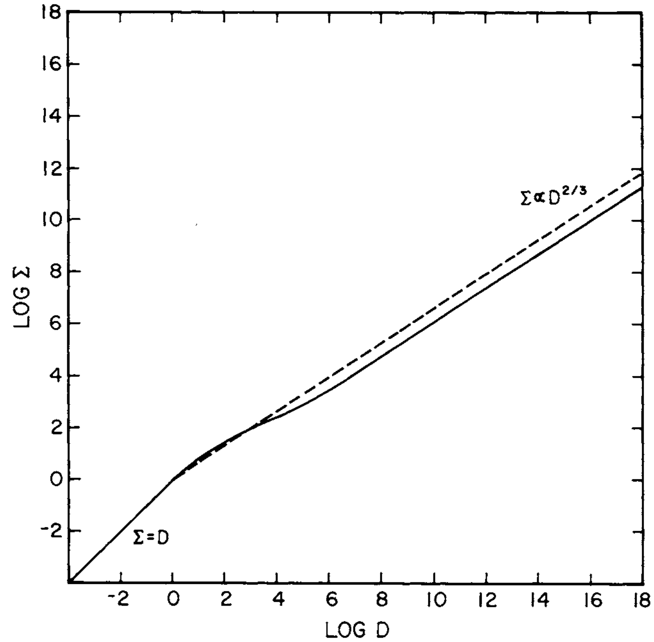

FIG. 1.—Σ vs. D (as defined in the text). The two theories of convection shown are MLT (*dashed line*) and FST (*solid line*).

or very opaque (large $\nabla_{\rm rad}$). Since very deep layers are nearly adiabatic ($\nabla \approx \nabla_{\rm ad}$), the difference between the two theories will have its greatest effect on ∇ in the superadiabatic transition region, where the opacity due to hydrogen ionization is very large. The solar models of Canuto & Mazzitelli (1991), which were computed with the two convection theories, show the expected effect. As those authors note, however, the effect is not enormous, and the near overlap of the two curves in our Figure 1, for all D, helps to explain why.

3. CONVECTIVE MIXING LENGTH

In both convection theories the convective mixing length, l, contains all of the compressibility effects. Since l is not known from first principles, it must be assumed. Most authors have taken l to be a multiple of the local pressure scale height,

$$l = \alpha_P H_P \,, \tag{7}$$

with α_P a constant of order unity. Canuto & Mazzitelli (1991, 1992) have recently proposed

$$l = \alpha_z z \,, \tag{8}$$

where z is the distance below the top of the convection zone and α_z is a variable coefficient of order unity. They recommended $\alpha_z = 1$. A similar assumption was made by Hofmeister & Weigert (1964) and Böhm & Stückl (1967), who set l equal to the distance to the nearest boundary of the convection zone if this distance was less than H_P. Other forms for a nonlocal mixing length have been proposed, e.g., l proportional to the actual density scale height (Schwarzschild 1961), l equal to the thickness of the strongly superadiabatic region (Böhm 1958) or of the whole convection zone (Latour 1970), and l equated to values found from fitting the analytical MLT equations to numerical results of two-dimensional simulations of stellar envelope convection (Deupree & Varner 1980).

It has long been common practice to infer an "average" value of α_P in the superadiabatic region by matching the effec-

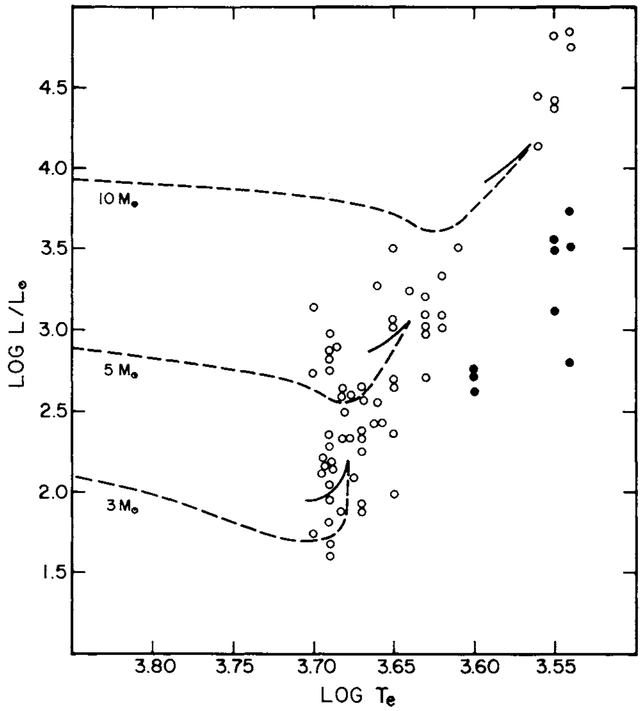

FIG. 2.—H-R diagram showing evolutionary tracks up to the second luminosity minimum on the red giant branch. Dashed segments represent very rapid stages. A best-fit value of α_P has been used for each track. Red giants and red supergiants in Galactic open clusters are shown as circles: *open circles*, core helium burning; *filled circles*, very late phases of evolution.

tive temperatures of appropriately computed stellar models to the observed effective temperatures of cool stars that are believed to contain significant convective envelopes. In the case of the Sun, acceptable models must satisfy the additional constraints of the Sun's known mass, luminosity, metals abundance, and age (Schwarzschild, Howard, & Härm 1957). Recent solar computations have indicated $\alpha_P \approx 1.9 \pm 0.3$ for the MLT model (Kim, Demarque, & Guenther 1991; Guenther et al. 1992; Lydon, Fox, & Sofia 1993; Sackmann, Boothroyd, & Kraemer 1993) if the calculated solar models include molecular sources in the total opacity. Numerical modeling of turbulent convection in solar-type convective envelopes suggests a mild depth dependence of α_P, with α_P falling in the range 1–2, if the numerically derived convective fluxes are fitted to the analytical MLT equations (Deupree & Varner 1980; Nordlund & Dravins 1990; Lydon, Fox, & Sofia 1992). Other stars have also been used as calibrators. The two solar-like components of α Centauri suggest $\alpha_P \approx 1.8$ (Lydon et al. 1993), while $\alpha_P \approx 1.5$ has been derived for Groombridge 1830 and various other old stars, both on and off the main sequence (VandenBerg 1983, 1985, 1992; VandenBerg & Bridges 1984; Bergbusch & VandenBerg 1992). It is fair to conclude that the MLT model requires $\alpha_P \approx 1.5$–2.1 for low-mass stars.

In the case of the FST model, only the choice $\alpha_z = 1$ has been tested. Nonetheless, this prescription works well for the very same types of stars (Canuto & Mazzitelli 1991, 1992; D'Antona et al. 1992; D'Antona & Mazzitelli 1994).

Our approach here is a similar one. We match computed stellar models to observed red giants and red supergiants in Galactic open clusters of somewhat younger age. Three different cases are considered: (1) MLT with $l = \alpha_P H_P$; (2) MLT with $l = \alpha_z z$; and (3) FST with $l = \alpha_z z$. It is assumed (as usual) that α remains constant along an evolutionary track. This assumption is a very good one here, because it applies principally to the short segment of track where the star slowly depletes core helium in the red region.

4. OBSERVATIONAL DATA

The open cluster data that we have used are contained in Table 6 of Harris (1976) and Table 6 of Stothers (1991). Harris's data are used for the intermediate-age clusters containing G and K giants and supergiants (and a few M giants). The somewhat older, metal-rich Hyades and Praesepe clusters are ignored here. Two stars are rejected from NGC 2546, following Sowell (1987), and two from NGC 3532 on the grounds of their apparently discrepant proper motions (Harris 1976). Otherwise, all membership class 1 and 2 stars with MK spectral classifications are accepted from Harris's list. For younger clusters containing M supergiants, the data tabulated by Stothers are used, but have been supplemented by Harris's data for the supergiants in NGC 457 and NGC 2439, even though the distances to these last two clusters are not very well determined.

Both Harris and Stothers assigned effective temperatures to the MK spectral classifications by using Johnson's (1966) empirical temperature scale. Comparison with the newer temperature scales of Lee (1970), Böhm-Vitense (1981), and Di Benedetto (1993) shows a very close interagreement among all four scales, the deviations around the mean never exceeding ± 0.01 in log T_e over the range of spectral types covered by the present sample of stars, G5 to M2. Since Johnson's scale lies close to the mean, we accept it here. (In doing so, we have corrected some minor errors in Harris's Table 6.)

The small spread of observed effective temperatures (Δ log $T_e \approx 0.05$) at a fixed luminosity in Figure 2 doubtless reflects, for the most part, evolution up and down the sloping red giant branch in the H-R diagram. Some of the observed spread may be due to a metallicity variation among the star clusters in the sample, but this effect should be minor, because the average metallicity of open clusters near the Sun is very close to solar (Nissen 1988; Clariá, Lapasset, & Minniti 1989; Cayrel de Strobel et al. 1992). Modest deviations from a solar metallicity would slightly alter the transformation relation between spectral type and effective temperature, and would also alter the actual effective temperatures of the red giants themselves; evolutionary tracks suggest that δ log $T_e \approx -0.06\ \delta$ log Z_e under the assumption that α is independent of metallicity (Stothers & Chin 1993). The rest of the scatter in Figure 2 is probably due to the effects of discreteness of the MK spectral classification system, the uncertainty of the cluster distance moduli, and the possible inclusion of some cluster nonmembers.

5. OTHER INPUT PHYSICS

Other physical input parameters include the following: initial hydrogen abundance by mass, $X_e = 0.700$; initial metals abundance by mass, $Z_e = 0.02$; intermediate-temperature opacities from Iglesias, Rogers, & Wilson (1992); high-temperature ($T > 10^8$ K) opacities from Cox & Stewart (1965, 1970); low-temperature ($T < 6 \times 10^3$ K) opacities, including diatomic and triatomic molecular sources, from Alexander, Johnson, & Rypma (1983), Alexander, Augason, & Johnson (1989), and Sharp (1992), in the form of a fitted formula published previously (Stothers & Chin 1993); and a gray radiative atmosphere, based on Eddington's (1926) approximation, for the layers above the photosphere. Convective overshooting, turbulent pressure, axial rotation, magnetic fields, and mass loss have been ignored.

Stellar masses in the present study are taken to be 3, 5, and 10 $M_\odot$. In the special case of 5 $M_\odot$, we have explored the sensitivity of the predicted red-giant effective temperatures to variations in some of the above physical input data. No significant shifts, defined as exceeding ± 0.01 in log T_e at a fixed luminosity, were found by making plausible changes in initial hydrogen abundance, initial iron abundance, photospheric pressure, or convective overshooting distance (either overshooting upward from the convective core or downward from the convective envelope). Our adopted equation of state, with ionization computed from the Saha equation, is believed to be quite accurate for the low-density conditions prevailing in red giant envelopes, and our neglect of the relatively ineffectual turbulent pressure is expected to generate little error (Henyey et al. 1965; Canuto & Mazzitelli 1991).

Larger shifts in effective temperature can be produced by changing the total initial metals abundance as well as the molecular contribution to the radiative opacities, as we have documented elsewhere (Stothers & Chin 1993). The resulting uncertainties are discussed in §§ 4 and 6.

6. NEW EVOLUTIONARY SEQUENCES

Evolutionary sequences of stellar models were computed from the zero-age main sequence into the region of yellow giants, and then up to the top and down to the bottom of the red giant branch. In Figure 2, the solid-line segments of evolutionary track represent slow, and hence easily observable, stages. Evolution beyond the second minimum on the red giant branch involves either a return up the red giant branch or a temporary looping out of the red giant region. There follows a final, rapid phase of stellar evolution, in which the stars of intermediate mass rise up to very bright luminosities and cool effective temperatures (e.g., Vassiliadis & Wood 1993). Such stars (filled circles in Fig. 2) are cooler than normal red giants and will be ignored in the present comparison; they can be readily identified in the original cluster H-R diagrams by their excessive luminosities.

Theoretical red giant branches have been calculated for various values of α and then compared with the observed cluster red giants. Since the theoretical luminosities are practically independent of α, we have simply matched predicted and observed effective temperatures at a fixed luminosity. A 2 σ error in the observed effective temperatures is estimated to be ± 0.02 in log T_e (§ 4). Since our models predict

$$\delta \log T_e \approx 0.3\, \delta \log \alpha\,, \tag{9}$$

the inferred values of α ought to be formally accurate to within 15%. As a strictly differential expression, equation (9) is probably very reliable, because it is essentially model-independent and represents well many other published red giant calculations (e.g., Henyey et al. 1965; Becker 1981).

Once all of the appropriate values of α have been determined, the evolutionary tracks can be intercompared at a fixed stellar mass. Not unexpectedly, these tracks look almost identical in the H-R diagram, which is the reason why only one track at each stellar mass is displayed in Figure 2.

We find for the best-fitting values of α those listed in the third column of Table 1. The most surprising feature in these results is that the MLT model implies a nonnegligible variation of α with stellar mass: as the mass increases, α_P drops or α_z rises. Previous theoretical studies, including our own, have routinely assigned a constant value of α_P, once α_P is fixed from, say, a study of the Sun. This is evidently not always a reliable procedure for MLT. On the other hand, the FST model with $\alpha_z = 1$ works as acceptably for red giants as it does for the Sun. This is all the more remarkable because, had we omitted molecular sources from the opacities, the predicted effective temperatures for red giants would have been hotter by ~ 0.05 in log T_e (Stothers & Chin 1993), which would have led to an apparent failure of the FST model (at least with the choice $\alpha_z = 1$).

Notice that, in all cases studied, $1 \le \alpha \le 3$ for both α_P and α_z. This lack of a very wide variation in α can be understood as follows. Close to the stellar surface, where the heat content and opacity of the gas are low, convection becomes inefficient in transporting the flux, and therefore $\nabla \approx \nabla_{\rm rad}$. As a result, the choice of l is unimportant there. At very deep layers, the efficiency of convection becomes so high that $\nabla \approx \nabla_{\rm ad}$ regardless of the value of l. Thus l makes a difference only in the intermediate transition layers. In these critical layers, $H_P \approx z$ (see the Appendix), and so the "effective" value of α_P within a deep convective envelope should be roughly of the same order as the "effective" value of α_z.

From Figure 1 it can be seen that the MLT and FST models yield roughly comparable values of $\nabla - \nabla_{\rm ad}$ for a fixed value of $\nabla_{\rm rad} - \nabla_{\rm ad}$. This lack of any wide divergence explains why the fitted values of α turn out to differ only moderately between the two convection models. Convection in the FST model, however, is somewhat less efficient, as illustrated in Figures 3 and 4 for a stellar model of 10 $M_\odot$ at the top of the red giant

TABLE 1

THEORETICAL RED GIANT BRANCHES

$M/M_\odot$	Convection Theory	l	Deepest $q_{\rm env}$	Red Top log $(L/L_\odot)$	Red Top log T_e	Second Red Bottom log $(L/L_\odot)$	Second Red Bottom log T_e
3	MLT	$2.8H_P$	0.170	2.20	3.68	1.95	3.70
	MLT	$1.7z$	0.171	2.20	3.67	1.95	3.70
	FST	$1.0z$	0.170	2.20	3.68	1.96	3.71
5	MLT	$2.1H_P$	0.207	3.05	3.64	2.86	3.67
	MLT	$2.0z$	0.207	3.05	3.64	2.86	3.67
	FST	$1.0z$	0.208	3.05	3.63	2.89	3.68
10	MLT	$1.8H_P$	0.226	4.15	3.56	3.92	3.59
	MLT	$2.2z$	0.227	4.15	3.56	3.93	3.59
	FST	$1.0z$	0.225	4.15	3.55	3.92	3.58

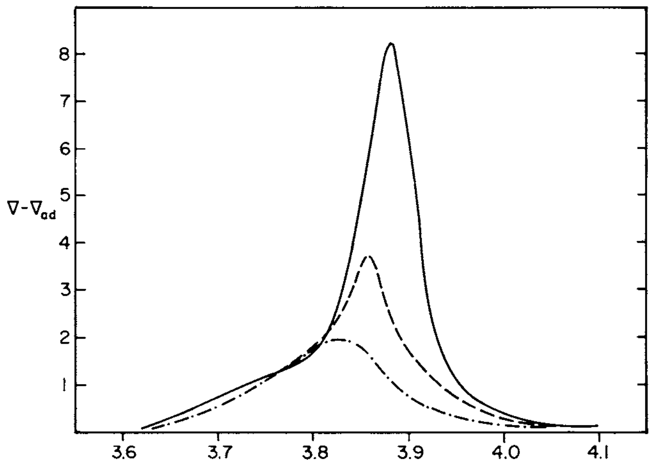

FIG. 3.—Superadiabatic temperature gradient vs. temperature in a stellar model of 10 $M_\odot$ at the top of the red giant branch. The three cases shown are: MLT with $l = 1.8H_P$ (*dash-dot line*); MLT with $l = 2.2z$ (*dashed line*); and FST with $l = 1.0z$ (*solid line*).

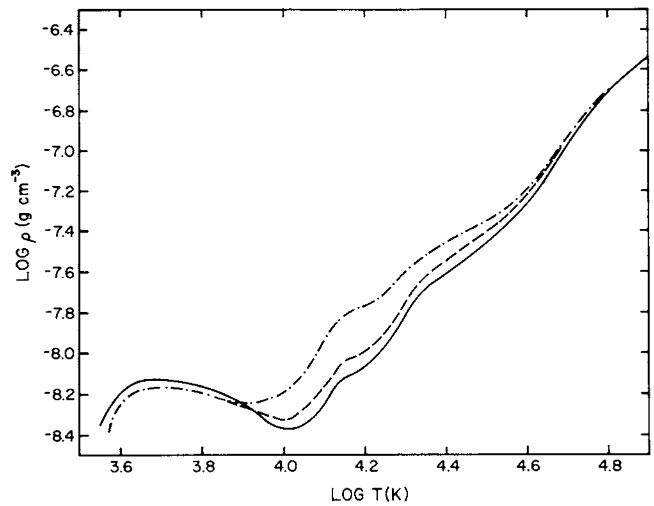

FIG. 4.—Density vs. temperature in a stellar model of 10 $M_\odot$ at the top of the red giant branch. Inflections in the curve occur where H, He, and He^+ become partially ionized. Same notation as for Fig. 3.

branch. Notice that the FST model produces a larger departure of ∇ from ∇_{ad} in the region of partial hydrogen ionization and, consequently, a slightly larger density inversion there. Equivalent results for the Sun were previously derived by Canuto & Mazzitelli (1991).

7. CONCLUSION

The MLT and FST models of stellar envelope convection can both reproduce the observed effective temperatures of Galactic red giants and red supergiants with appropriate choices of α. Somewhat surprisingly, the more extensive comparison with observations presented here reveals a significant dependence of α on stellar mass in the case of MLT. Although any inferred absolute value of α will be uncertain because of uncertainties in many other physical quantities, the signs of the gradients of α_P and α_z with stellar mass are probably correct. These derived trends now reopen the question of whether α_P is possibly smaller in brighter stars along the very extended red giant branch of old star clusters (Chieffi & Straniero 1989; Demarque, Green, & Guenther 1992).

In contrast to the situation for MLT, the FST model with a constant $\alpha_z = 1$ is very successful for all cases where it has been tested. Since it satisfies also the necessary solar constraints (Canuto & Mazzitelli 1991), its economy of assumptions and its better physical justification seem to commend it over MLT. One might nevertheless question the physical validity of the Ansatz $l = z$. Lydon et al. (1992) have objected that numerical simulations of turbulent convection show l of the order of H_P at all depths (Chan & Sofia 1987; Nordlund & Dravins 1990). However, since H_P is nearly equal to z in the region of efficient superadiabatic stellar convection, their objection may well be unimportant. Moreover, our fits to stellar observations do not contradict the assumption $l = z$.

From the point of view of obtaining reliable evolutionary sequences up to the end of core helium burning, there is no clear preference for one convection theory over the other. Table 1 and Figure 4 demonstrate how insensitive the structure of the deep radiative interior of the star is to the choice of convection theory for the outer envelope. The base of the outer convection zone at its greatest depth in mass fraction, q_{env} (Table 1) and the temperature and density there (Fig. 4) show little variation. As a check, we have continued further our evolutionary calculations for a star of 10 $M_\odot$ and have determined that the subsequent blue loops on the H-R diagram are essentially identical for the three parameter choices of Table 1. Under some circumstances, the choice of convective mixing length can influence whether a blue loop is triggered or suppressed (Chin & Stothers 1991), but the deep structure of the star underneath the expanding or contracting envelope always remains virtually unaffected, as has long been known (Schwarzschild 1958).

Other possible tests of the two convection theories—along the lines of the present work—can be made for the following classes of stars: the reddest superluminous stars on the asymptotic giant branch in Galactic clusters, the most luminous red supergiants in Galactic associations, the metal-poor luminous red supergiants in the Small Magellanic Cloud, and the massive blue supergiants that contain large iron convection zones.

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APPENDIX

The local pressure scale height, H_P, can be expressed in an approximate analytical form as a function of the geometrical depth, z, below the upper boundary of the outer convection zone. We assume that the convection zone has the structure of a polytrope of index n, and we adopt the equation of state of an ideal gas. The basic equations are then

$$\frac{d \ln P}{dz} = \frac{1}{H_P} = \frac{GM}{(R - z)^2} \frac{\rho}{P}, \tag{A1}$$

$$\frac{d\ln T}{d\ln P} = \frac{1}{n+1}\,, \qquad P = \frac{k}{\mu m_p}\rho T\,. \tag{A2}$$

Expanding the gravitational acceleration to lowest order in z/R, we integrate these equations for $T(z)$ and evaluate H_P, finding

$$H_P = H_0 + \left(\frac{1}{n+1} - \frac{2kT_0 R}{\mu m_p GM}\right)z\,, \tag{A3}$$

where a zero subscript designates the upper boundary. The second term in the parentheses is approximately equal to 4 (T_0/T_c) $(\langle\rho\rangle/\rho_c)$ (Schwarzschild 1958, p. 32) and so can be neglected. Consequently, $H_P = H_0 + z/(n+1)$. Near the upper boundary, $H_P \approx H_0$, while deep in the convection zone $H_P \approx z/(n+1)$, which is a result identical to the one derived by Lamb (1932) for a polytropic atmosphere of constant gravity. Canuto & Mazzitelli (1992) also noted that $H_P \sim z$.

Superadiabatic layers of the envelope are typically characterized by a shallow density inversion in the region of partial hydrogen ionization, just above where the adiabatic region begins. As a result, the density in these layers does not increase inward very much from the photospheric value (see, for example, the solar models of Canuto & Mazzitelli 1991). Therefore, a value of $n \approx 0$ is appropriate. In this case, $H_P \approx z$ inside the transition region.

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